

Integrable deformations of Hamiltonian systems and q -symmetries ¹

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Abstract

The complete integrability of the hyperbolic Gaudin Hamiltonian and other related integrable systems is shown to be easily derived by taking into account their $sl(2, \mathbb{R})$ coalgebra symmetry. By using the properties induced by such a coalgebra structure, it can be proven that the introduction of any quantum deformation of the $sl(2, \mathbb{R})$ algebra will provide an integrable deformation for such systems. In particular, the Gaudin Hamiltonian arising from the non-standard quantum deformation of the $sl(2, \mathbb{R})$ Poisson algebra is presented, including the explicit expressions for its integrals of motion. A completely integrable system of nonlinearly coupled oscillators derived from this deformation is also introduced.

1 Introduction: on q -Poisson coalgebras

Let us consider the $sl(2)$ Poisson-Lie algebra

$$\{J_3, J_{\pm}\} = \pm 2 J_{\pm}, \quad \{J_-, J_+\} = \alpha J_3. \quad (1.1)$$

(where α is a real positive parameter and the usual $\alpha = 1$ case can be easily get by performing the change of basis $J_{\pm} \rightarrow J_{\pm}/\sqrt{\alpha}$). The Casimir function for this algebra reads

$$C(J_3, J_{\pm}) = \frac{\alpha}{4} J_3^2 - J_+ J_-. \quad (1.2)$$

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The Poisson algebra (1.1) is endowed with a coalgebra structure [1, 2] by means of the “primitive” coproduct $\Delta : sl(2) \rightarrow sl(2) \otimes sl(2)$ defined as follows:

$$\Delta(J_3) = J_3 \otimes 1 + 1 \otimes J_3, \quad \Delta(J_{\pm}) = J_{\pm} \otimes 1 + 1 \otimes J_{\pm}. \quad (1.3)$$

Compatibility between (1.3) and (1.1) means that Δ is a Poisson map: the three functions defined through (1.3) close also a Poisson $sl(2)$ algebra. The map (1.3) can be introduced in a similar manner for any Lie algebra. The coalgebra structure, together with the known relevance of Lie algebras in the context of Liouville integrability [3, 4], are the keystones of the universal construction of integrable systems introduced in [5] that we shall summarize in the next Section.

From that point of view, the search for q -Poisson algebras can be seen as the problem of finding deformations of a given algebra endowed with compatible deformations of the coproduct (1.3) preserving it as a Poisson map. In fact, two relevant and distinct structures deforming $sl(2, \mathbb{R})$ appeared in quantum group literature during last years (see, for instance, [6]), and can be realized as q -Poisson algebras as follows:

a) The “*standard*” deformation [7, 8] $sl_q(2)$, which is given by the following deformed Poisson brackets ($q = e^z$):

$$\{J_3, J_+\} = 2J_+, \quad \{J_3, J_-\} = -2J_-, \quad \{J_-, J_+\} = \frac{\sinh z J_3}{z}, \quad (1.4)$$

which are compatible with the deformed coproduct

$$\begin{aligned} \Delta(J_3) &= J_3 \otimes 1 + 1 \otimes J_3, \\ \Delta(J_+) &= J_+ \otimes e^{\frac{z}{2}J_3} + e^{-\frac{z}{2}J_3} \otimes J_+, \\ \Delta(J_-) &= J_- \otimes e^{\frac{z}{2}J_3} + e^{-\frac{z}{2}J_3} \otimes J_-, \end{aligned} \quad (1.5)$$

in the sense that that (1.5) is a Poisson algebra homomorphism with respect to (1.4). The function

$$\mathcal{C}_z^{(s)}(J_3, J_{\pm}) = \left(\frac{\sinh(\frac{z}{2}J_3)}{z} \right)^2 - J_+ J_-, \quad (1.6)$$

is the deformed Casimir for this Poisson coalgebra.

b) The “*non-standard*” quantum $sl(2, \mathbb{R})$ deformation [9, 10] is defined by:

$$\begin{aligned} \Delta(J_-) &= J_- \otimes 1 + 1 \otimes J_-, \\ \Delta(J_+) &= J_+ \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_+, \\ \Delta(J_3) &= J_3 \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_3, \end{aligned} \quad (1.7)$$

$$\{J_3, J_+\} = 2J_+ \cosh z J_-, \quad \{J_3, J_-\} = -2 \frac{\sinh z J_-}{z}, \quad \{J_-, J_+\} = 4J_3. \quad (1.8)$$

The deformed Casimir reads

$$\mathcal{C}_z^{(n)}(J_3, J_\pm) = J_3^2 - \frac{\sinh z J_-}{z} J_+. \quad (1.9)$$

The aim of this contribution is to present an application of this latter q -Poisson algebra in the construction of an integrable deformation of the Gaudin Hamiltonian. This deformation can be also interpreted in the context of a chain of nonlinearly coupled oscillators or, equivalently, in relation with a certain integrable perturbation of the motion of a particle under any central potential in the N -dimensional Euclidean space. Through these examples we will show an intrinsic connection between quantum deformations and nonlinear interactions depending on the momenta. Results concerning the standard deformation of the Gaudin-Calogero system [11] can be found in [5, 12].

2 The formalism

By following [5], we can state the following result: any coalgebra (A, Δ) with Casimir element C can be considered as the generating symmetry of a large family of integrable systems. We shall consider here classical mechanical systems only and, consequently, we shall make use of Poisson realizations D of the algebra A of the form $D : A \rightarrow C^\infty(q, p)$. However, we recall that the formalism is also directly applicable to quantum mechanical systems. The constructive procedure is as follows.

Let (A, Δ) be a (Poisson) coalgebra with generators X_i ($i = 1, \dots, l$) and Casimir function $C(X_1, \dots, X_l)$. This means that the coproduct $\Delta : A \rightarrow A \otimes A$ is a Poisson map. Let us consider the N -th coproduct map $\Delta^{(N)}$

$$\Delta^{(N)} : A \rightarrow A \otimes A \otimes \dots^N \otimes A, \quad (2.1)$$

which is obtained (see [5]) by applying recursively the two-coproduct $\Delta^{(2)} \equiv \Delta$ in the form

$$\Delta^{(N)} := (id \otimes id \otimes \dots^{N-2} \otimes id \otimes \Delta^{(2)}) \circ \Delta^{(N-1)}. \quad (2.2)$$

By taking into account that the m -th coproduct ($m \leq N$) of the Casimir $\Delta^{(m)}(C)$ can be embedded into the tensor product of N copies of A as

$$\Delta^{(m)} : A \rightarrow \{A \otimes A \otimes \dots^m \otimes A\} \otimes \{1 \otimes 1 \otimes \dots^{N-m} \otimes 1\}, \quad (2.3)$$

it can be shown that,

$$\{\Delta^{(m)}(C), \Delta^{(N)}(X_i)\}_{A \otimes A \otimes \dots^N \otimes A} = 0, \quad i = 1, \dots, l, \quad 1 \leq m \leq N. \quad (2.4)$$

With this in mind it can be proven [5] that, if \mathcal{H} is an *arbitrary* (smooth) function of the generators of A , the N -particle Hamiltonian defined on $A \otimes A \otimes \dots^N \otimes A$ as the N -th coproduct of \mathcal{H}

$$H^{(N)} := \Delta^{(N)}(\mathcal{H}(X_1, \dots, X_l)) = \mathcal{H}(\Delta^{(N)}(X_1), \dots, \Delta^{(N)}(X_l)), \quad (2.5)$$

fulfils

$$\{C^{(m)}, H^{(N)}\}_{A \otimes A \otimes \dots \otimes A} = 0, \quad 1 \leq m \leq N, \quad (2.6)$$

where the N functions $C^{(m)}$ ($m = 1, \dots, N$) are defined through the m -th coproducts of the Casimir C

$$C^{(m)} := \Delta^{(m)}(C(X_1, \dots, X_l)) = C(\Delta^{(m)}(X_1), \dots, \Delta^{(m)}(X_l)), \quad (2.7)$$

and all the integrals of motion $C^{(m)}$ are in involution

$$\{C^{(m)}, C^{(n)}\} = 0, \quad \forall m, n = 1, \dots, N. \quad (2.8)$$

Therefore, provided a non-trivial realization of A on a one-particle phase space is given, the N -particle Hamiltonian $H^{(N)}$ will be a function of N canonical pairs (q_i, p_i) and is, by construction, completely integrable with respect to the ordinary Poisson bracket

$$\{f, g\} = \sum_{i=1}^N \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right). \quad (2.9)$$

Moreover, its constants of motion will be given by the $C^{(m)}$ functions, all of them functionally independent since each of them depends on the first m pairs (q_i, p_i) of canonical coordinates. Note that with such one-particle realizations the first Casimir $C^{(1)}$ will be a number, and we are left with $N - 1$ constants of motion with respect to $H^{(N)}$.

Let us stress now that the previous construction is valid for quantum algebras with no extra assumptions. Quantum algebras are also (deformed) coalgebras (A_z, Δ_z) , and any function of the generators of a given quantum algebra with Casimir element C_z will provide, under a chosen deformed representation, a completely integrable Hamiltonian. Therefore, the obtention of quantum algebras is a direct method to get integrable deformations of those Hamiltonian systems with underlying coalgebra symmetry.

3 Oscillator chains from $sl(2, \mathbb{R})$ coalgebras

It is well-known that $sl(2, \mathbb{R})$ can be considered as a dynamical algebra for the one-dimensional harmonic oscillator. As we shall see in the sequel, the $sl(2, \mathbb{R})$ coalgebra will give us the complete set of integrals of the motion of a chain of N independent harmonic oscillators with the same frequency ω . When the non-standard quantum deformation is considered, a new integrable oscillator chain with long range interactions depending on the momenta is obtained.

3.1 The non-deformed case

Let us consider the $sl(2, \mathbb{R})$ coalgebra (1.1) and (1.2) with $\alpha = 4$. A one-particle phase space realization of this Poisson algebra with vanishing Casimir is given by the functions:

$$f_-^{(1)} = D(J_-) = q_1^2, \quad f_+^{(1)} = D(J_+) = p_1^2, \quad f_3^{(1)} = D(J_3) = q_1 p_1. \quad (3.1)$$

From it, the harmonic oscillator Hamiltonian is recovered if the following linear function of the generators of $sl(2, \mathbb{R})$

$$\mathcal{H} = J_+ + \omega^2 J_-, \quad (3.2)$$

is represented through (3.1):

$$H^{(1)} = D(\mathcal{H}) = p_1^2 + \omega^2 q_1^2. \quad (3.3)$$

Now, the representation $(D \otimes D)$ when applied onto the primitive coproduct (1.3), leads to the following two-particle phase space realization of $sl(2, \mathbb{R})$

$$f_-^{(2)} = q_1^2 + q_2^2, \quad f_+^{(2)} = p_1^2 + p_2^2, \quad f_3^{(2)} = q_1 p_1 + q_2 p_2. \quad (3.4)$$

that, in turn, gives rise to the uncoupled oscillator Hamiltonian:

$$H^{(2)} = (D \otimes D)(\Delta^{(2)}(\mathcal{H})) = f_+^{(2)} + \omega^2 f_-^{(2)} = p_1^2 + p_2^2 + \omega^2(q_1^2 + q_2^2). \quad (3.5)$$

Note that now the frequency of both oscillators is the same. The phase space realization of the coproduct of the Casimir will give us the corresponding integral of motion

$$C^{(2)} = (D \otimes D)(\Delta^{(2)}(C)) = -(q_1 p_2 - q_2 p_1)^2, \quad (3.6)$$

that turns out to be the square of the angular momentum, as expected.

The construction of the m -dimensional functions from the m -th coproduct is straightforwardly obtained by the induction (2.2):

$$f_-^{(m)} = \sum_{i=1}^m q_i^2, \quad f_+^{(m)} = \sum_{i=1}^m p_i^2, \quad f_3^{(m)} = \sum_{i=1}^m q_i p_i. \quad (3.7)$$

From it, the uncoupled chain of N harmonic oscillators (all of them with the same frequency) is obtained as the representation of the same dynamical Hamiltonian (3.2):

$$H^{(N)} = f_+^{(N)} + \omega^2 f_-^{(N)} = \sum_{i=1}^N (p_i^2 + \omega^2 q_i^2) \quad (3.8)$$

together with the integrals of motion ($m = 2, \dots, N$), that are deduced from the m -th coproducts of the Casimir and are shown to be

$$C^{(m)} = - \sum_{i < j}^m (q_i p_j - q_j p_i)^2. \quad (3.9)$$

Note that the integrals $C^{(m)}$ given by the $sl(2, \mathbb{R})$ coalgebra are just the quadratic Casimirs of the $so(m)$ algebras with $m = 2, \dots, N$. It is also well known that the Hamiltonian (3.8) is $so(N)$ invariant, since it can be interpreted as the one for a particle moving on the N -dimensional Euclidean space under the potential $\omega^2 r^2$.

3.2 An integrable deformation from $U_z(sl(2, \mathbb{R}))$

Now, we introduce a one-particle deformed phase space realization of (1.8):

$$\begin{aligned} f_-^{(1)} &= D_z(J_-) = q_1^2, \\ f_+^{(1)} &= D_z(J_+) = \frac{\sinh z q_1^2}{z q_1^2} p_1^2, \\ f_3^{(1)} &= D_z(J_3) = \frac{\sinh z q_1^2}{z q_1} p_1. \end{aligned} \quad (3.10)$$

This realization is also characterized by a vanishing deformed Casimir function (1.9)

Let us consider again the dynamical generator $\mathcal{H} = J_+ + \omega^2 J_-$. Under (3.10), we obtain the Hamiltonian

$$H_z^{(1)} = D_z(\mathcal{H}) = \frac{\sinh z q_1^2}{z q_1^2} p_1^2 + \omega^2 q_1^2. \quad (3.11)$$

The corresponding two-particle phase space realization of $U_z(sl(2, \mathbb{R}))$ is obtained from both the coproduct (1.7) and (3.10):

$$\begin{aligned} f_-^{(2)} &= q_1^2 + q_2^2, \\ f_+^{(2)} &= \frac{\sinh z q_1^2}{z q_1^2} p_1^2 e^{z q_2^2} + \frac{\sinh z q_2^2}{z q_2^2} p_2^2 e^{-z q_1^2}, \\ f_3^{(2)} &= \frac{\sinh z q_1^2}{z q_1} p_1 e^{z q_2^2} + \frac{\sinh z q_2^2}{z q_2} p_2 e^{-z q_1^2}. \end{aligned} \quad (3.12)$$

The associated two-particle Hamiltonian is

$$H_z^{(2)} = \frac{\sinh z q_1^2}{z q_1^2} p_1^2 e^{z q_2^2} + \frac{\sinh z q_2^2}{z q_2^2} p_2^2 e^{-z q_1^2} + \omega^2 (q_1^2 + q_2^2), \quad (3.13)$$

and the deformed coproduct for the deformed Casimir (1.9) reads

$$C_z^{(2)} = - \frac{\sinh z q_1^2 \sinh z q_2^2}{z^2 q_1^2 q_2^2} (q_1 p_2 - q_2 p_1)^2 e^{-z q_1^2} e^{z q_2^2}. \quad (3.14)$$

If we rewrite (3.13) in the form

$$\begin{aligned} H_z^{(2)} &= H^{(2)} + p_1^2 \left(\frac{\sinh z q_1^2}{z q_1^2} e^{z q_2^2} - 1 \right) + p_2^2 \left(\frac{\sinh z q_2^2}{z q_2^2} e^{-z q_1^2} - 1 \right) \\ &= H^{(2)} + z (p_1^2 q_2^2 - p_2^2 q_1^2) + o[z^2], \end{aligned} \quad (3.15)$$

the nature of the interaction introduced by the non-standard deformation with respect to (3.5) can be appreciated. Note that the series expansion (3.15) can be meaningful when the deformation parameter z is small, and it should be explored in order to analyse the dynamics.

The N -dimensional generalization for this system can be derived from the m -th order coproducts (2.2) induced from (1.7):

$$\begin{aligned} f_-^{(m)} &= \sum_{i=1}^m q_i^2, \\ f_+^{(m)} &= \sum_{i=1}^m \frac{\sinh z q_i^2}{z q_i^2} p_i^2 e^{z K_i^{(m)}(q^2)}, \\ f_3^{(m)} &= \sum_{i=1}^m \frac{\sinh z q_i^2}{z q_i} p_i e^{z K_i^{(m)}(q^2)}, \end{aligned} \quad (3.16)$$

where the K -functions are defined as:

$$K_i^{(m)}(x) = - \sum_{k=1}^{i-1} x_k + \sum_{l=i+1}^m x_l, \quad (3.17)$$

$$\begin{aligned} K_{ij}^{(m)}(x) &= K_i^{(m)}(x) + K_j^{(m)}(x) \\ &= -(x_i - x_j) - 2 \sum_{k=1}^{i-1} x_k + 2 \sum_{l=j+1}^m x_l, \quad i < j. \end{aligned} \quad (3.18)$$

Therefore, the N -dimensional Hamiltonian is just $H_z^{(N)} = f_+^{(N)} + \omega^2 f_-^{(N)}$ and the following constants of motion are deduced:

$$C^{(m)} = - \sum_{i < j}^m \frac{\sinh z q_i^2 \sinh z q_j^2}{z^2 q_i^2 q_j^2} (q_i p_j - q_j p_i)^2 e^{z K_{ij}^{(m)}(q^2)}. \quad (3.19)$$

Throughout all the computations leading to (3.19), the following property becomes useful:

$$\frac{\sinh(z \sum_{i=1}^m x_i)}{z} = \sum_{i=1}^m \frac{\sinh z x_i}{z} e^{z K_i^{(m)}(x)}. \quad (3.20)$$

Note that under the limit $z \rightarrow 0$ we recover all the expressions presented in the previous paragraph.

3.3 Anharmonic chains and their deformation

We can now consider a more general dynamical Hamiltonian \mathcal{H} of the form

$$\mathcal{H} = J_+ + \mathcal{F}(J_-), \quad (3.21)$$

where $\mathcal{F}(J_-)$ is an arbitrary smooth function of J_- . The formalism summarized in Section 2 ensures that the corresponding system arising from the N -th coproduct of (3.21) is completely integrable, with \mathcal{H} being *any* function of the coalgebra generators. Explicitly, this means that any N -particle Hamiltonian of the form

$$H^{(N)} = f_+^{(N)} + \mathcal{F}(f_-^{(N)}) = \sum_{i=1}^N p_i^2 + \mathcal{F}\left(\sum_{i=1}^N q_i^2\right), \quad (3.22)$$

is completely integrable, and (3.9) are its constants of motion. Obviously, the integrability (in fact, superintegrability) of (3.22) is a well-known result, since (3.22) is just the Hamiltonian describing the motion of a particle in a N -dimensional Euclidean space under the action of a central potential. In terms of oscillator chains, the linear case $\mathcal{F}(J_-) = \omega^2 J_-$ leads to the previous harmonic case, and the quadratic one $\mathcal{F}(J_-) = J_-^2$ would give us an interacting chain of quartic oscillators. Further definitions of the function \mathcal{F} would give us many other anharmonic chains, all of them sharing the same dynamical symmetry and the same integrals of the motion.

Let us now stress that, by using the non-standard quantum $sl(2, \mathbb{R})$ algebra, it is clear that a realization of (3.21) in terms of (3.16) gives us:

$$\begin{aligned} H_z^{(N)} &= \sum_{i=1}^N \frac{\sinh z q_i^2}{z q_i^2} p_i^2 e^{z K_i^{(N)}(q^2)} + \mathcal{F}\left(\sum_{i=1}^N q_i^2\right) \\ &= \sum_{i=1}^N p_i^2 + \mathcal{F}\left(\sum_{i=1}^N q_i^2\right) + \sum_{i=1}^N p_i^2 \left(\frac{\sinh z q_i^2}{z q_i^2} e^{z K_i^{(N)}(q^2)} - 1\right), \end{aligned} \quad (3.23)$$

which is an integrable deformation of (3.22) with (3.19) being again the associated integrals. This result can be interpreted as a perturbation of the original anharmonic chain through long-range interacting terms depending on the momenta.

4 The Gaudin Hamiltonian and $sl(2, \mathbb{R})$ coalgebras

If we substitute the canonical realizations used until now in terms of angular momentum realizations of the same abstract $sl(2, \mathbb{R})$ Poisson coalgebra, the very same construction will lead us to a long-range interacting “classical spin chain” of the Gaudin type on which the quantum deformation can be easily implemented.

In particular, let us consider the classical angular momentum realization S

$$g_3^{(1)} = S(J_3) = \sigma_3^1, \quad g_+^{(1)} = S(J_+) = \sigma_+^1, \quad g_-^{(1)} = S(J_-) = \sigma_-^1, \quad (4.1)$$

where the variables σ_i^1 fulfil

$$\{\sigma_3, \sigma_+\} = 2\sigma_+, \quad \{\sigma_3, \sigma_-\} = -2\sigma_-, \quad \{\sigma_-, \sigma_+\} = 4\sigma_3, \quad (4.2)$$

and are constrained by a given constant value of the Casimir function (1.2) in the form $c_1 = (\sigma_3^1)^2 - \sigma_-^1 \sigma_+^1$.

As usual, m different copies of (4.1) (that, in principle, could have different values c_i of the Casimir) can be distinguished with the aid of a superscript σ_l^i . Then, the m -th coproduct provides the following realization of the non-deformed $sl(2, \mathbb{R})$ Poisson coalgebra :

$$g_l^{(m)} = (S \otimes \dots \otimes S)(\Delta^{(m)}(\sigma_l)) = \sum_{i=1}^m \sigma_l^i, \quad l = +, -, 3. \quad (4.3)$$

Now, we can apply the usual construction and take \mathcal{H} from (3.2). As a consequence, the uncoupled oscillator chain (3.8) is equivalent to

$$H^{(N)} = g_+^{(N)} + \omega^2 g_-^{(N)} = \sum_{i=1}^m (\sigma_+^i + \omega^2 \sigma_-^i), \quad (4.4)$$

and the Casimirs $C^{(m)}$ read ($m = 2, \dots, N$):

$$C^{(m)} = (g_3^{(m)})^2 - g_-^{(m)} g_+^{(m)} = \sum_{i=1}^m c_i + \sum_{i < j}^m (\sigma_3^i \sigma_3^j - \sigma_-^i \sigma_+^j - \sigma_-^j \sigma_+^i). \quad (4.5)$$

Note that these Casimirs are just Gaudin Hamiltonians of the hyperbolic type [13, 14] (in fact, we shall consider $C^{(N)}$ as the one defining a general Gaudin magnet).

The implementation of a non-standard deformation in the Gaudin system is now straightforward. The deformed angular momentum realization corresponding to the non-standard deformation $U_z(sl(2, \mathbb{R}))$ is:

$$\begin{aligned} g_-^{(1)} &= S_z(J_-) = \sigma_-^1, \\ g_+^{(1)} &= S_z(J_+) = \frac{\sinh z \sigma_-^1}{z \sigma_-^1} \sigma_+^1, \\ g_3^{(1)} &= S_z(J_3) = \frac{\sinh z \sigma_-^1}{z \sigma_-^1} \sigma_3^1, \end{aligned} \quad (4.6)$$

where the classical coordinates σ_l^1 are defined on the cone $c_1 = (\sigma_3^1)^2 - \sigma_-^1 \sigma_+^1 = 0$ (therefore, we are considering the zero realization).

It is easy to check that the m -th order of the coproduct (1.7) in the above representation leads to the following functions

$$g_-^{(m)} = \sum_{i=1}^m \sigma_-^i,$$

$$\begin{aligned}
g_+^{(m)} &= \sum_{i=1}^m \frac{\sinh z\sigma_-^i}{z\sigma_-^i} \sigma_+^i e^{zK_i^{(m)}(\sigma_-)}, \\
g_3^{(m)} &= \sum_{i=1}^m \frac{\sinh z\sigma_-^i}{z\sigma_-^i} \sigma_3^i e^{zK_i^{(m)}(\sigma_-)},
\end{aligned} \tag{4.7}$$

that define the non-standard deformation of (4.4). Therefore, the following “non-standard Gaudin Hamiltonians” are obtained

$$\begin{aligned}
C_z^{(m)} &= (g_3^{(m)})^2 - \frac{\sinh z g_-^{(m)}}{z} g_+^{(m)} \\
&= \sum_{i=1}^m \left(\frac{\sinh z\sigma_-^i}{z\sigma_-^i} \right)^2 e^{2zK_i^{(m)}(\sigma_-)} \{ (\sigma_3^i)^2 - \sigma_-^i \sigma_+^i \} \\
&\quad + \sum_{i < j}^m \frac{\sinh z\sigma_-^i \sinh z\sigma_-^j}{z^2 \sigma_-^i \sigma_-^j} e^{zK_{ij}^{(m)}(\sigma_-)} (\sigma_3^i \sigma_3^j - \sigma_-^i \sigma_+^j - \sigma_+^i \sigma_-^j). \tag{4.8}
\end{aligned}$$

Since we are working in the zero representation with $(\sigma_3^i)^2 - \sigma_-^i \sigma_+^i = 0$, the expression (4.8) can be simplified

$$C_z^{(m)} = \sum_{i < j}^m \frac{\sinh z\sigma_-^i \sinh z\sigma_-^j}{z^2 \sigma_-^i \sigma_-^j} e^{zK_{ij}^{(m)}(\sigma_-)} (\sigma_3^i \sigma_3^j - \sigma_-^i \sigma_+^j - \sigma_+^i \sigma_-^j). \tag{4.9}$$

These integrals are the angular momentum counterparts to (3.19). Note that, as it was found for the standard case in [5], the deformation can be interpreted as the introduction of a variable range exchange [15] in the model (compare (4.9) with (4.5)).

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